One of the fascinating things about calculus is that we believe that it was 'invented' by two different people, i.e., *Gottfried Wilhelm Leibniz* and *Isaac Newton*. However, that does not mean that the two methods are identical; both may be used for the same calculations, but their foundation and the justification for the art of calculus are very different. Thus, the ensuing task would be to investigate the foundations of each and understand the philosophical and practical implication of the differences. This essay will try to see if we can discover some of these differences by comparing Leibniz's characteristic triangle with Newton's Lemma 7.

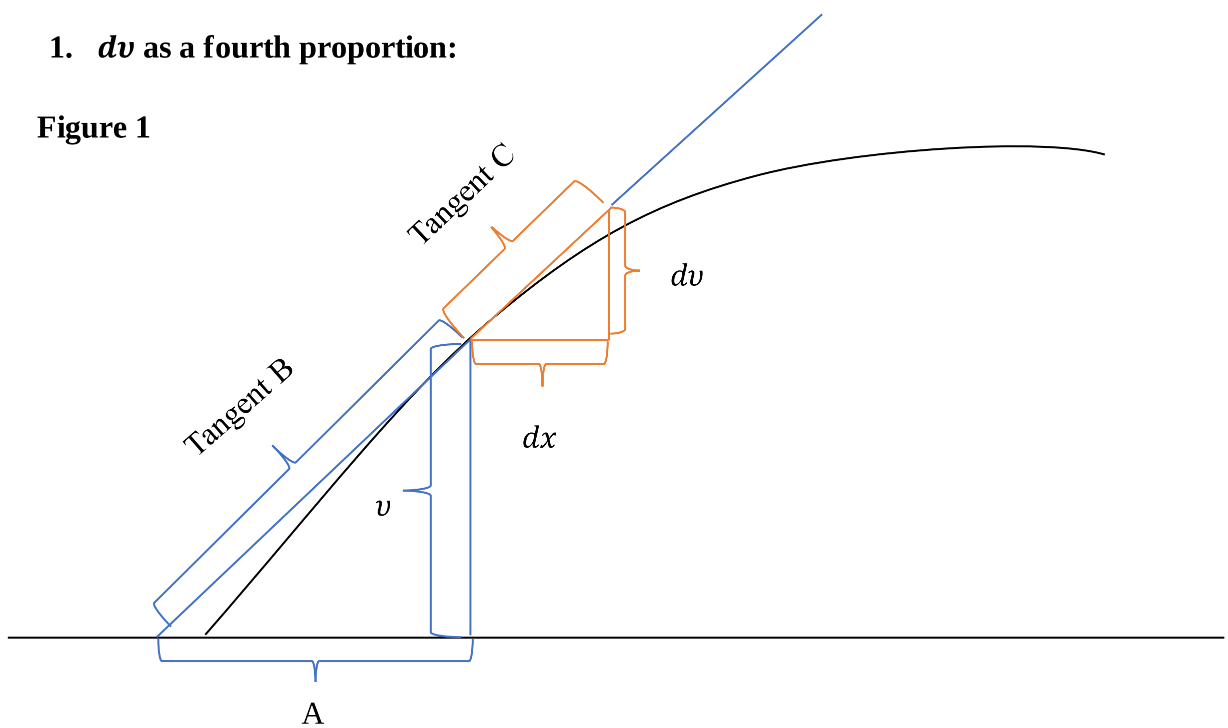
Let us start by taking a look at Leibniz's characteristic triangle. According to the 'Junior Mathematics Manual,'[i] [ii] the difference of the ordinate (dv) can be understood in two ways:

1. dv can be understood as the fourth term of a proportion.
2. dv can be understood as the differences of the v's (thus 'truly' of the ordinates).

So, let us see what these two different ways of understanding dv indicate about the nature of dv by looking at the figure below:

1. dv as a fourth proportion:

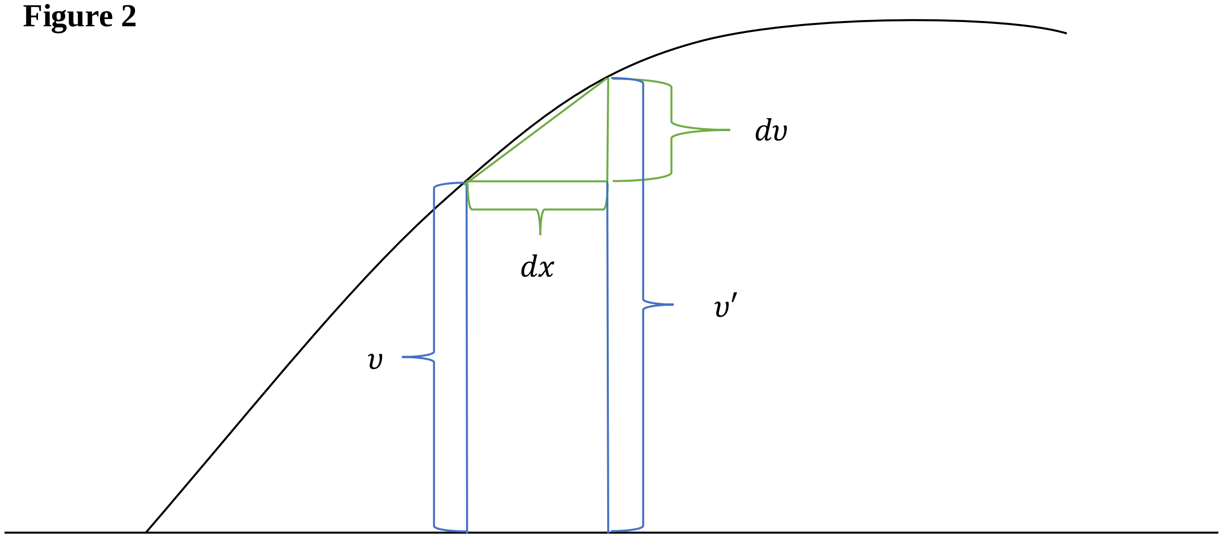
Figure 1



These two triangles (A, v, tangent B and dx, dv, tangent C) are similar and thus proportional to each other. In other words, $A : v :: dx : dv$. By using proportionality, we can understand the relationship of the infinitesimals by means of the proportional relationship with the finite. However, there is a problem with this way of understanding; dv does not represent the difference of the ordinate (for it is the length from the tangent to dx and not from the curve to dx). Thus, because it is not touching the curve, it is hard to relate it as a difference of the ordinate of the curve. To solve that problem, we have resort to the other way of understanding dv.

2. dv as the difference of the v's:

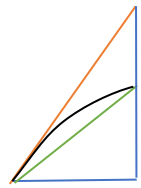
Figure 2



In this case, $dv = v' - v$ and indeed corresponds to the curve and the difference of the ordinates. However, now we are incapable of knowing anything about dv since we cannot understand the indefinitely small through the finite.

Leibniz, however, found a way to make these two ways of thinking compatible. To do that, we need a way to make the hypotenuse of the second triangle consistent with the hypotenuse in the first triangle (the tangent). We can do that if we take the arbitrary dx as an infinitely small line. Then it follows that v and v' are 'in a certain sense' two copies of itself, or in other words: *a difference of v itself.* (see manual p. 22) Furthermore, as Leibniz himself points out (see manual p. 12), taking two points as infinitely close on a curve amounts to equating it to a polygon with infinitely many sides: the hypothenuse of the triangles coincide and are equivalent with the curve itself:

Figure 3



Infinitely small curvilinear triangle = infinitely small rectilinear triangles

Leibniz calls this triangle, where the 'tangential' triangle (figure 1), the 'cordial' triangle (figure 2), and the curvilinear triangle are equal, the *characteristic triangle*.

However, I believe that Leibniz can only justify his definition of the characteristic triangle by so-called indivisibles--magnitudes that do not follow Euclid's proposition X.1.[1iii] It seems to me that, unlike Newton, it is not the case for Leibniz that the tangent, chord, and arc will vanish when they coincide; <u>for Leibniz sees a curve as a polygon with infinitely many sides, or as Galileo says: "a continuum out of *indivisible* atoms."</u>[iv] Thus, I think that Leibniz must believe in the existence of indivisible atoms to justify his characteristic triangle. For further evidence that he really believes in these indivisibles (and not just assumes their existence for practical usage), I would like to point out that Leibniz's philosophy expressed in the essays "Principles of Nature and Grace, Based on Reason" and "The Monadology" seems to reveal Leibniz's belief in these indivisible entities. He explains in these essays the nature of the so-called 'monad.' He calls a monad 'the true atom of nature,' but it is an 'atom' that has no parts, extension, shape, nor divisibility.[2v] Thus, it seems natural for Leibniz to have applied his belief in the existence of such indivisible things to his mathematics. Just as we have seen, the difference of the ordinates is a difference in itself, which deprives it of extension and likewise, the hypothenuse of the triangle has no shape or extension since it is equivalent with a curved line and a straight line. Also, his curve, like Galileo's, is composed of a continuum of indivisible atoms (like a body is composed of an infinite aggregate of monads)[3]. As a result, I feel myself compelled to boldly claim that Leibniz uses indivisibles to support his theory on calculus.

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[1] *"Two unequal magnitudes being set out if from greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process is repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out."*

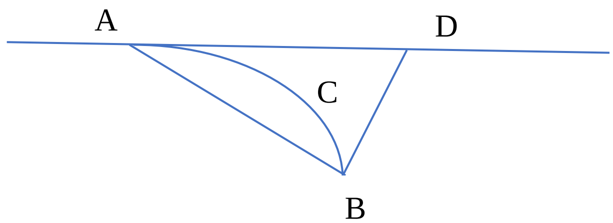
[2] See Monadology paragraph 3.

[3]. To get an understanding of such a composition, see Leibniz's essay Principles of Nature and Grace, Based on Reason paragraph 3.

Now we have examined some of Leibniz's foundations, let us take a look at Newton's foundations. But in order to understand Lemma 7 of the Principia,[vi] [vii] let us first take a quick look at a summarized version of Newton's proof of Lemma 6:

Figure 4

Legend:
AD = tangent
ACB = continuous curvature
AB = chord

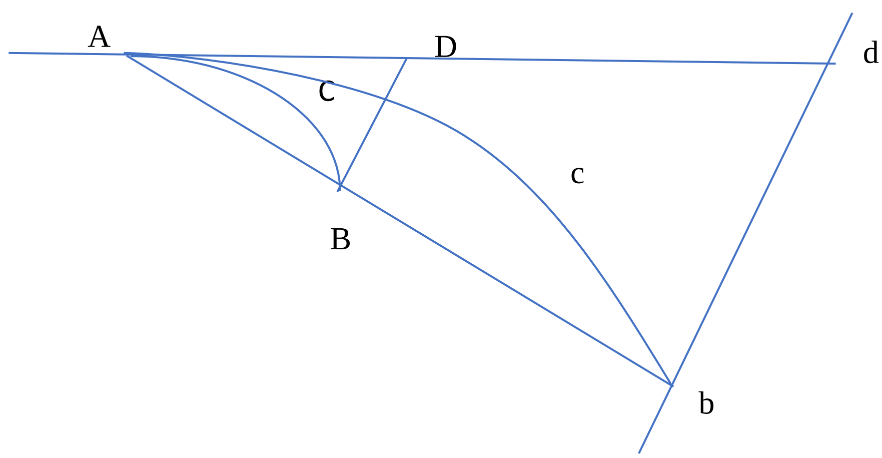


Newton claims that when the points A and B approach each other and coincide, angle BAD is diminished in infinitum and *ultimately vanishes*. For, if that angle does not disappear, the arc ACB will contain with the tangent AD a curvilinear angle (CAD) that is equal to the rectilinear angle (BAD), which renders the curve discontinuous. Hence, Newton's definition of continuity is that there is a unique tangent for every point where no line can fall in between, i.e., it is not possible that the rectilinear angle of the tangent and the chord is equivalent to the curvilinear angle of the tangent and curve: the rectilinear angle will stop existing (vanish) before such an occurrence. For Newton, a curve can thus *not* be composed of infinitely many infinitely small sides, i.e., he *cannot* see a curve in the same way as Leibniz: a polygon with infinite many sides. Hence, now the question arises why Newton's calculus does *not* require indivisibles as was the case with Leibniz's characteristic triangle.

For the answer to our question, let us now continue to Lemma 7:

Figure 5

Legend:
AD = tangent
ACB = continuous curvature
AB = chord
Ad = produced tangent
Ab= produced chord
Acb = arc that is similar to ACB
bd = parallel with BD

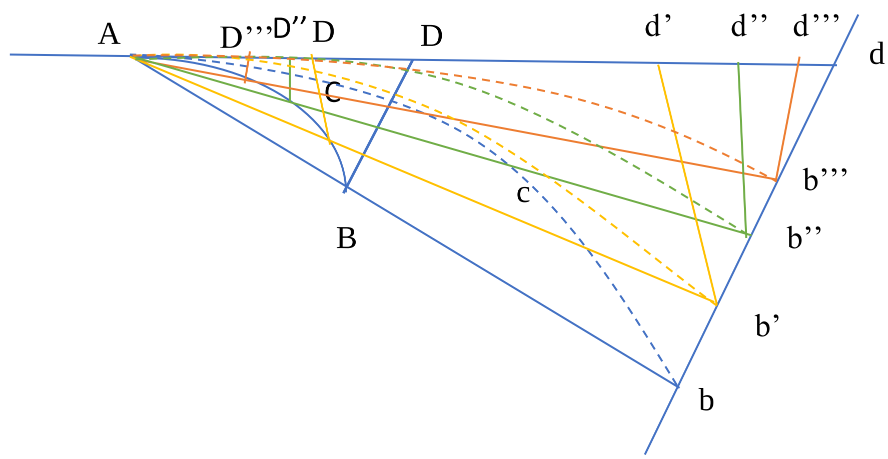


Newton proves in this Lemma that the arc, chord, and tangent are in the *ultimate ratio of equality* with each other. To make us understand what he means by that, he incorporated a method (somewhat) similar to Leibniz's characteristic triangle: the 'produced triangle' (see triangle Abd in the figure). This triangle allows us to visualize the result of A and B approaching each other and when they come together. This triangle is always proportional to the smaller triangle ABD and is produced to the line with points b and d, and its arc stays always similar to the arc of triangle ABD. I understand that it is hard to see that the small triangle always stays in proportion with the produced triangle; hence I think it is useful to give the following figure:

Figure 6

Instances where the yellow, green, and orange line hit the small arc, are respectively B', B'', B'''.

The dashed arcs represent the arcs that are supposed to be similar to ACB; it is difficult to represent, but the important thing to see is that the arcs become closer and closer to a straight line.



As you can see in figure 6, the finite triangle always maintains its proportionality to the decreasing triangle, and the decreasing arc and finite arc always stay similar. We are therefore allowed 'to translate' what happens when point A and B come closer and when they eventually coincide. Through Lemma 6 we know that the angle BAD (=angle bAd) will ultimately vanish. It is evident to see that the finite lines Ab and Ad will then coincide, but also the arc Acb which lies between them will coincide, i.e., Ab = Ad = Acb.

However, we cannot say that ultimately AB = AD = ACB, because when A and B come together, these lines have no magnitude anymore. That is the reason why Newton speaks of an *ultimate ratio of equality* and **not** of *ultimate equality of* *magnitudes*. So ultimately the following ratio occurs: $AB : Ab :: AD : Ad :: ACB : Acb :: 1 : 1$, hence $AB : AD :: 1 : 1$ or $AB : ACB :: 1 : 1$. However, this is not an expression of magnitudes, causing a new kind of problem that Leibniz did not have with his indivisibles.

For, what does this ultimate ratio expresses for Newton? And how can he justify violating Euclid's definition of a ratio?[4] In the Scholium to the Lemmas, Newton tries to clarify his understanding of ultimate ratios: "By the ultimate ratio of evanescent quantities is to be understood the ratio of quantities, *not* before they vanish, *not* afterward, <u>but [that] with which they vanish</u>…Those ultimate ratios with which the quantities vanish are *not* truly ratios of ultimate quantities, <u>but limits to which the ratios of quantities decreasing with limit always approach</u>, and which they can attain no more nearly than by any given difference you please, but never go beyond, nor arrive at before the quantities are diminished in infinitum."[5] So, Newton sees the ratio of equality as *the limit* which will be attained at the moment of vanishing.

But, as stated asked above, how can Newton justify seeing a ratio not as a magnitude but as a limit, for it is against the Euclidean definition of ratios. I think that Newton's reply will be: *"I am capable of determining this limit geometrical; thus, it is perfectly fine to use ratios."* My inspiration for this answer comes from the same scholium mentioned above, for there he says: "since this limit is certain and definite, it is really a *geometrical problem* to determine it." And Lemma 7 indeed shows how he determines this limit geometrically, for the finite triangle always makes us able to understand the infinite in the form of spatial magnitudes. Hence, the validity of this ultimate ratio of equality is still founded on its relevance to an extended Euclidean object. So, it does *(arguably)* correspond to Euclidean geometry and, instead of violating it, perfects it.

Thus, by looking at Leibniz's 'characteristic triangle' and Newton's 'produced triangle,' I believe that we can make an argument that Leibniz's calculus uses so-called indivisibles and that Newton tries to avoid the philosophical obscurities of indivisibles by basing his calculus on an ultimate ratio of limits. So far, it seems to me that its practical use does not differ so much despite the philosophical differences. With both definitions, the hypothenuse, chord, and arc find equality in a certain sense, and both can express this relationship in the finite. But we should not neglect the philosophical differences which influence the way how we interpret the results of each operation. Moreover, the philosophical difference might have a

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[4] Euclid book V, definition 3: *"A ratio is a sort of relation in respect of size between two magnitudes of the same kind."*

[5] The italics and underlines are added.

relevance to the practical applications of each calculus that is not visible at the surface, a surface that is, in my opinion, worth diving into.[6]

[i] Lasell, Brendon. *"Notes on Leibniz's A New Method" Reading and Notes for Junior Mathematics*, edited by Brendon Lasell. St. John's College, 2018.

[ii] Leibniz, Gottfried Wilhelm. *A New Method for Finding Maxima and Minima*. Actis Erud. Lips. Oct. 1684. p. 467-473

[iii] Euclides, et al. *Euclids Elements*. Green Lion Press, 2013.

[iv] Galilei, Galileo. *Two New Sciences*. Wisconsin U.P., 1974.

[v] Leibniz, Gottfried Wilhelm, et al. *Philosophical Essays*. Hackett Publishing Company, 1989.

[vi] Newton, Isaac. *Philosophiæ Naturalis Principia Mathematica*. Watchmaker, 2010.

[vii] Densmore, Dana, and William H. Donahue. *Newtons Principia: The Central Argument;* Green Lion Press, 2003.

[6] One practical difference caused by their different philosophical points of view might be found in their notation. For Leibniz can represent dy as a quantity that stands alone (for example: instead of $\frac{dy}{dx} = 2$, he can notate it as $dy = 2dx$), but can Newton do such a thing as well? The answer seems to me no, because dy stands then for an indivisible quantity.